# Solution for Hw 9



Ex 16.7: 8)

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -z+\frac{1}{2+x} & \tan^{-1} y & x+\frac{1}{4+z} \end{vmatrix} = (0,-2,0)$$

$$f(x,y,z) = 4x^2+y+z^2 \Rightarrow \nabla f = (8x,1,2z)$$

$$\vec{p} = (0,1,0) \Rightarrow |\nabla f\cdot p| = 1$$

$$\therefore \iint_S \nabla\times\vec{F}\cdot n\,dS = \iint_R (0,-2,0)\cdot\frac{(8x,1,2z)}{|\nabla f|}\,\frac{|\nabla f|}{(1)}\,dA$$

$$= \iint_R -2\,dA = -2(\text{Area of } R) = -4\pi$$

12) Let $C$ be the common boundary of $S_1$ & $S_2$, with orientation induced by the outward normal of $S_1$.

Then $$\iint_S \nabla\times\vec{F}\cdot\vec{n}\,dS = \iint_{S_1}\nabla\times\vec{F}\cdot\vec{n}\,dS + \iint_{S_2}\nabla\times\vec{F}\cdot\vec{n}\,dS$$

$$= \oint_C \vec{F}\cdot d\vec{r} + \oint_{-C}\vec{F}\cdot d\vec{r}$$

$$= 0$$

16)

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x-y & y-z & z-x \end{vmatrix} = (1,1,1)$$

$$r_r\times r_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & -1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (r\cos\theta, r\sin\theta, r)$$

$$\therefore \iint_S \nabla\times\vec{E}\cdot\vec{n}\,dS = \int_0^{2\pi}\int_0^5 (r\cos\theta, r\sin\theta, r)\cdot(1,1,1)\,dr\,d\theta$$

$$= 25\pi$$

20) $F = \nabla f = \left( \frac{-x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right)$

a) let $\gamma(t) = (a\cos t, a\sin t, 0)$, $t \in [0, 2\pi]$.

$\gamma'(t) = (-a\sin t, a\cos t, 0)$

$$\Rightarrow \vec{F} \cdot \vec{\gamma}' = \left( -\frac{a\cos t}{a^3}, \frac{-a\sin t}{a^3}, 0 \right) \cdot (-a\sin t, a\cos t, 0)$$

$$= 0$$

$$\Rightarrow \oint_C \vec{F}\, d\vec{\gamma} = 0 \,_{//}$$

b) $$\oint_C \vec{F} \cdot d\vec{\gamma} = \iint_S \nabla \times \vec{F} \cdot n\, dS = \iint_S \nabla \times \nabla f \cdot n\, dS = 0 \,_{//}$$

26) $P_y = N_z = M_z = P_x = 0$,

$N_x = \frac{y^2 - x^2}{(x^2+y^2)^2} = M_y$

$\Rightarrow \operatorname{curl} \vec{F} = 0$.

However, let $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$.

Then $$\oint \vec{F} \cdot d\vec{\gamma} = \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = 2\pi \neq 0 \,_{//}$$

Additional and Advanced Exercise:

12) a) Partition the sheet into small pieces.

Let $\Delta_i\sigma$ be the area of $i$th piece.

Let $(x_i, y_i, z_i)$ be a point on the $i$th piece.

Then the mass $\approx x_i y_i \Delta_i\sigma$.

$\therefore$ Work done by gravity in moving the $i$th piece to $xy$-plane $\approx (g x_i y_i \Delta_i\sigma) z_i = g x_i y_i z_i \Delta_i\sigma$.

$$\Rightarrow \text{Total Work done} = \iint_S g\, xyz\, dS \,_{//}$$

b) $\iint_S gxyz\,dS = g\iint_R xy(1-x-y)\sqrt{1^2+(-1)^2+(-1)^2}\,dA$

$= \sqrt{3}g\int_0^1\int_0^{1-x}(xy-x^2y-xy^2)\,dy\,dx = \sqrt{3}g\int_0^1\left(\frac{1}{6}x-\frac{1}{2}x^2+\frac{1}{2}x^3-\frac{1}{6}x^4\right)$

$dx = \sqrt{3}g\left[\frac{1}{12}x^2-\frac{1}{6}x^3+\frac{1}{8}x^4-\frac{1}{30}x^5\right]_0^1 = \frac{\sqrt{3}g}{120}$ //

c) The required work done

$= gM\bar{z} = gM\left(\frac{M_{xy}}{M}\right) = gM_{xy} = \iint_S gxyz\,d\sigma = \frac{\sqrt{3}g}{120}$ //

14) $S$ is given by $z=\sqrt{x^2+y^2}$, $z\in[1,2]$.
Partition $S$ into small pieces
Let $\Delta_i\sigma$ be the area of the $i$th piece.
Magnitude of the force acting on the $i$th piece due to liquid pressure $\approx w(2-z_i)\Delta_i\sigma$.
$\Rightarrow$ The total force $= \iint_S w(2-z)\,dS$

$= \iint_R w(2-\sqrt{x^2+y^2})\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,dA$

$= \iint_R \sqrt{2}w(2-\sqrt{x^2+y^2})\,dA = \int_0^{2\pi}\int_1^2 \sqrt{2}w(2-r)r\,dr\,d\theta$

$= \int_0^{2\pi}\frac{2\sqrt{2}w}{3}\,d\theta = \frac{4\sqrt{2}\pi w}{3}$ //

16) By Gauss's law, $\iint_S \vec{F}\cdot\vec{n}\,dS = 4\pi GmM$.

If $\vec{F}=\nabla\times\vec{H}$, by Q12), Ex16.7, $\iint_S \vec{F}\cdot\vec{n}\,dS = 0$, contradiction.

So $\vec{F} \neq \nabla \times \vec{H}$.

$\boxed{\text{Rmk: You } \underline{\text{CANNOT}} \text{ use divergence thm to prove it as } \vec{F} \text{ is not well-defined at the origin here.}}$

18) Given that $\nabla \times F_1 = \nabla \times F_2$.

$$\Rightarrow \nabla \times (F_1 - F_2) = 0 \Rightarrow \exists f \text{ s.t. } F_1 - F_2 = \nabla f.$$

As $\nabla \cdot F_1 = \nabla \cdot F_2 \Rightarrow \nabla \cdot \nabla f = \nabla \cdot (F_1 - F_2) = 0 \Rightarrow \Delta f = 0$.

Also, on the surface $S$, $\nabla f \cdot n = (F_2 - F_1) \cdot n = 0$.

Since $\nabla \cdot (f \nabla f) = \nabla f \cdot \nabla f + f \Delta f$, by div. thm,

$$\iiint_D |\nabla f|^2 dV + \iiint_D f \Delta f \, dV = \iint_S f \nabla f \cdot n \, dS = 0.$$

$$\Rightarrow \iiint_D |\nabla f|^2 dV = 0 \Rightarrow \vec{F_2} - \vec{F_1} = 0 \Rightarrow F_1 = F_2 \; {}_{//}$$

21) $\vec{r}(x,y,z) = (x,y,z) \Rightarrow \nabla \cdot \vec{r} = 1+1+1 = 3$.

$$\iiint_D \nabla \cdot \vec{r} \, dV = 3 \iiint_D dV = 3V.$$

$$\text{So } V = \frac{1}{3} \iiint_D \nabla \cdot \vec{r} \, dV \overset{\text{div.}}{\underset{\text{thm.}}{=}} \frac{1}{3} \iint_S \vec{r} \cdot \vec{n} \, dS \; {}_{//}$$

## Practice Problems: Additional and Advanced Exercise

11) a) Partition the string into pieces.
Let $\Delta_i s$ be the length of the $i$th piece.
Let $(x_i, y_i)$ be a point on it.
The work done by gravity in moving the $i$th piece to $x$-axis $\approx (g x_i y_i \Delta_i s) y_i = g x_i y_i^2 \Delta_i s$

$$\Rightarrow \text{Total work done} = \int_C g x y^2 \, ds$$

b) Work $= \int_C g x y^2 ds = \int_0^{\frac{\pi}{2}} g(2\cos t)(4\sin^2 t)(2) dt = \frac{16}{3} g$ //

c) $W = \left(\int_C g x y\, ds\right) \left( \frac{\int_C g x y^2 ds}{\int_C g x y\, ds} \right) = \frac{16}{3} g$ //

13) a) Partition the sphere into small pieces.
Let $\Delta \sigma_i$ be the surface area of the ith piece.
Let $(x_i, y_i, z_i)$ be a point on it.
Magnitude of the force due to the pressure $\approx w(4 - z_i)\Delta\sigma_i$
$\Rightarrow$ Total magnitude $= \iint_S w(4-z) dS$.

b) Force on the ith piece $\approx w(4-z)(-n) = w(z-4) \cdot n$.
Vertical component $= w(z-4) \cdot \vec{n} \cdot \vec{k}$.
$\Rightarrow$ Total buoyant force $= \iint_S w(z-4) \vec{k} \cdot \vec{n}\, dS$.

c) By div. thm, $\iint_S w(z-4)\vec{k} \cdot \vec{n}\, dS = \iiint_D \nabla \cdot (w(z-4)\vec{k}) dV$
$= \iiint_D w\, dV = w(\text{volume of the ball}) = \frac{4}{3}\pi w$ //

15) By Stoke's thm, $\oint_C \vec{E} \cdot d\vec{r} = \iint_S (\nabla \times \vec{E}) \cdot \vec{n}\, dS$
$= \iint_S \left(-\frac{\partial \vec{B}}{\partial t}\right) \cdot \vec{n}\, d\sigma = -\frac{\partial}{\partial t} \iint_S \vec{B} \cdot \vec{n}\, dS$ //

17) $\oint_C f \nabla g \cdot dr = \iint_S \nabla \times (f \nabla g) \cdot n\, dS$ (Stoke's thm)
$= \iint_S (f \nabla \times \nabla g + \nabla f \times \nabla g) \cdot \vec{n}\, dS$

$$= \iint_S \left( f\,\overrightarrow{(0,0,0)} + \nabla f \times \nabla g \right) \cdot \vec{n}\, dS$$

$$= \iint_S (\nabla f \times \nabla g) \cdot \vec{n}\, dS_{\text{4}}$$

19) False.

Take $\vec{F} = (y, x) \neq 0$.

$$\nabla \cdot \vec{F} = 0, \quad \nabla \times F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ y & x & 0 \end{vmatrix} = (0,0,0).$$

20) $|r_u \times r_v|^2 = |r_u|^2 |r_v|^2 \sin^2\theta = |r_u|^2 |r_v|^2 (1 - \cos^2\theta)$

$= |r_u|^2 |r_v|^2 - |r_u|^2 |r_v|^2 \cos^2\theta = |r_u|^2 |r_v|^2 - (r_u \cdot r_v)^2.$

$\Rightarrow |r_u \times r_v|^2 = EG - F^2$

$\Rightarrow dS = |r_u \times r_v|^2 du\,dv = \sqrt{EG - F^2}\, du\,dv$